3/MTH–201 **Syllabus–2023**

2 0 2 4

(December)

FYUP : 3rd Semester Examination

MAJOR

MATHEMATICS

(Group Theory)

MTH–201

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

UNIT—1

1. *(a)* Let G be the set of all real 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad - bc \neq 0$ is a rational number. Prove that G forms a group under matrix multiplication. 5

   (b) Let G be a group. Show that for all $a, b \in G$, $(a^{-1})^{-1} = a$ and $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$. 2+3=5

(c) Let G be a group, $a \in G$ and $b \in G$. Let n be a positive integer. Show that $(aba^{-1})^n = aba^{-1}$, if and only if, $b = b^n$. 4

(d) If G is a finite group, then show that there exists a positive integer N such that $a^N = e$ for all $a \in G$. 4

2. (a) If G is a group of even order, then prove that it has an element $a \neq e$ satisfying $a^2 = e$. 4

(b) Show that the set $\{1, -1, i, -i\}$ forms a group with respect to multiplication. Is this group an abelian group? Justify. 5

(c) Let G be a group and $a, b \in G$. Show that the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y in G. 4

(d) Let G be a group. Show that the identity element of G is unique. Also, show that every $a \in G$ has a unique inverse in G. 2+3=5

UNIT—2

3. (a) Show that every permutation can be expressed as a product of disjoint cycles. 6

(b) Express (1, 2, 3) (4, 5) (1, 6, 7, 8, 9) (1, 5) as the product of disjoint 2 cycles. 3

(c) Let A_n be the set of even permutations in S_n. Show that A_n forms a group of order $\frac{n!}{2}$. 5

(d) Let $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Define a product in G by $a^4 = e$, $b^2 = a^2$, $ba = a^{-1}b$. Show that G forms a group (called the quaternion group). 5

4. (a) Determine which of the following permutations are even : 3

(i) (1 2 3 4 5) (1 2 3) (4 5)

(ii) (1 2 3) (4 5) (1 6 7 8)

(b) Compute $a^{-1}ba$, where $a = (1, 3, 5)(1, 2)$ and $b = (1, 5, 7, 9)$. 3

(c) Let

$$D_{2n} = \{e, a, a^2, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}$$

be a set of $2n$ elements. Define the product in D_{2n} by the relations $a^n = e$, $b^2 = e$ and $ab = ba^{-1}$.

(i) Find the form of the product $(x^i y^j)(x^k y^l)$ as $x^\alpha y^\beta$. 3

(ii) Using this, prove that G forms a group. 5

(iii) If $n = 4$, then show that D_8 is the group of symmetries of the square. 5

UNIT—3

5. (a) State and prove Lagrange's theorem. 4

(b) Let H be a subgroup of a group G, $a \in G$ and $aHa^{-1} = \{aha^{-1} \mid h \in H\}$. Show that aHa^{-1} is a subgroup of G. Also, show that if H is finite, then $o(H) = o(aHa^{-1})$. 3+3=6

(c) Prove that any subgroup of a cyclic group is itself a cyclic group. 4

(d) Let $G = \langle a \rangle$ be a cyclic group of order 8 and let $H = \langle a^4 \rangle$. Find all the cosets of H in G. 3

(e) Define the centre Z of a group G and show that Z is a subgroup of G. 2

6. (a) State and prove Euler's theorem. 4

(b) Show that every group of prime order is cyclic. 5

(c) Let G be a finite cyclic group of order n. Show that G has $\phi(n)$ generators. 5

(d) If H is a subgroup of G, and $N(H) = \{a \in G \mid aHa^{-1} = H\}$, then prove that $N(H)$ is a subgroup of G and $H \subseteq N(H)$. 3+2=5

UNIT—4

7. (a) If N and M are normal subgroups of G, then prove that NM is also a normal subgroup of G. 5

(b) If H and K are normal subgroups of a group G such that $H \subseteq K$, then show that

$$\frac{G}{K} \cong \frac{(G/H)}{(K/H)}$$ 6

(c) Let G be a group and ϕ an automorphism of G. If $a \in G$ is of order $o(a) > 0$, then show that $o(\phi(a)) = o(a)$. 4

(d) Let G be a group, N a normal subgroup of G, T an automorphism of G. Prove that $T(N)$ is a normal subgroup of G. 4

8. (a) Show that the group $I(G)$ of inner automorphisms of G is isomorphic to $\frac{G}{Z(G)}$. 5

(b) Let H and K be normal subgroups of G such that $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H$, $k \in K$. 4

(c) Let G be a group, H and K subgroups of G such that K is normal in G. Then show that $H \cap K$ is normal in H and the quotient group $\frac{H}{H \cap K}$ is isomorphic to the quotient group $\frac{HK}{K}$. 1+4=5

(d) Let G be the group of non-zero complex numbers under multiplication and let N be the set of complex numbers of absolute value 1. (That is $a + bi \in N$ if $a^2 + b^2 = 1$). Show that $\frac{G}{N}$ is isomorphic to the group of all positive real numbers under multiplication. 5

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